Kochen-Specker Theorem and Games

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December 22, 2007

1 Introduction

One of the statements one would hear pretty early in any quantum mechanics course is that for quantum particles there are properties which do not have definite values. For example, if we measure $|+\rangle$ in standard basis, the best we can do to describe the outcome is to say, that with probability 1/2 we will get 0 and with probability 1/2 we will get 1. Yet one could wonder, is this really the way things are or is it just because our description of this quantum particle is incomplete, but in fact there are some *hidden variables* inside every quantum system that contain enough information to tell the outcome of measurement (see Fig. 1).

However, many people disdain the idea of hidden variables, so there has been rather long history of theorems that forbid theories involving hidden variables. Usually if one wants to prove some no hidden variables theorem he has to assume something about the hidden variables theories he is going to forbid. One of such assumptions is non-contextuality, which means that the outcome we get after measuring with an observable is the same regardless of what other observables we choose to measure at the same time.

The first theorem forbidding hidden variables theories is due to von Neumann in 1932. Then for a third of a century everyone cited von Neumann's work carelessly, till Bell in 1966 indicated that there was an unreasonable assumption in von Neumann's proof and urged for further examination of the problem (see [6] and [1] for more historical details). Bell also constructed a hidden variables model for one qubit that could assign the outcome of any observable in a way consistent with quantum mechanics. He also proved two theorems that forbid non-contextual hidden variables theories using counterexamples with larger quantum systems. Yet surprisingly he himself thought that hidden variables do exists and considered these theorems only as characterizations of the hidden variables theory we should look for.

In this essay we will discuss several versions of one of the two Bell's theorems. The theorem we will consider was also independently proven by Kochen and Specker in 1967 (see [5]), hence it is called Bell-Kochen-Specker theorem. For shortness and in order not to mix this theorem with another Bell's theorem from now on we will refer to it as Kochen-Specker theorem. We will be interested in proving several versions of this theorem and the following lemma will turn out to be of great use:

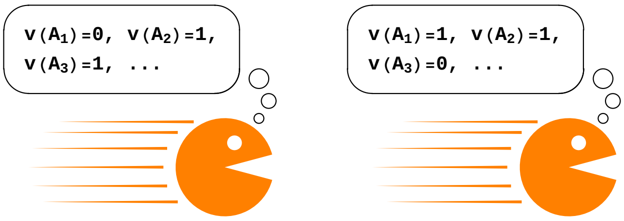

Figure 1: Quantum particles with hidden variables, where $v(A_i)$ stands for the outcome of observable A_i we are going to get upon measuring the particle. Note that we have assumed non-contextuality.

Lemma 1. If mutually commuting observables $A_1, A_2, \ldots, A_n$ satisfy some functional identity
$$f(A_1, A_2, \ldots, A_n) = 0,$$
then the outcomes we get will satisfy the same functional identity. Therefore, the values $v(A_i)$ assigned to these observables in an individual system must also be related by
$$f\Big(v(A_1), v(A_2), \ldots, v(A_n)\Big) = 0$$

Proof. Further on we will apply this lemma only for polynomial functional identities. Therefore, we will prove only the case when f is some polynomial. Mutually commuting observables $A_1, A_2, \ldots, A_n$ can be simultaneously diagonalized in some basis $\mathcal{B}$ (see [8] pp. 172). Thus, mathematically we can think of measuring with $A_1, A_2, \ldots, A_n$ as measuring in basis $\mathcal{B}$. If $|b_i\rangle \in \mathcal{B}$ is the state after the measurement, then we got outcome λ_i for observable A_i, where $A_i |b_i\rangle = \lambda_i |b_i\rangle$. So, we have

$$0 = \langle b_i| f\Big(v(A_1), v(A_2), \ldots, v(A_n)\Big) |b_i\rangle = f(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

which means that the functional identity f is satisfied by the outcomes of the measurement. □

2 Kochen-Specker theorem

Theorem 1. *(3-dimensional version of Kochen-Specker theorem)* In a Hilbert space of dimension ≥ 3 there is a set of observables for which it is impossible to assign outcomes in a way consistent with quantum mechanics formalism (i.e., in a way that all functional identities satisfied by mutually commuting observables are also satisfied by the values assigned to them in each individual system).

We will not give a complete proof of this theorem. Instead, we will give the outline of the proof and a bit discuss various constructions that people have used to prove this theorem. Main idea of the proof can be stated as follows. First, we find an explicit set of observables and some functional identities that are satisfied by some mutually commuting observables form this set. Then we show that these functional identities cannot be satisfied by the outcomes assigned to the observables.

The set of observables we are interested in will consist of squares of the observables measuring the spin component along some direction in real space. Let us now examine these observables a bit closer. The observables for measuring the spin component in the directions of x, y and z axes are as follows:

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The observables for measuring the spin component in an arbitrary direction $n = (x, y, z) \in S^2$ is given by:

$$S_n = n \cdot S = xS_x + yS_y + zS_z,$$

where dot stands for inner product. One can check that the eigenvalues of S_n are 1, -1 and 0 (or see some book on quantum mechanics e.g. [4]). Therefore the eigenvalues of S_n^2 are 1 and 0.

Claim 1. If unit vectors v_1, v_2 are orthogonal, then $S_{v_1}^2$ commutes with $S_{v_2}^2$.

Proof. It is just arithmetic to check that S_x, S_y, S_z are mutually commuting. Then let $v_i = (a_{i,1}, a_{i,2}, a_{i,3})$ and $S_1 := S_x, S_2 := S_y, S_3 := S_z$. Now we have

$$[S_{v_1}, S_{v_2}] = S_{v_1}S_{v_2} - S_{v_2}S_{v_1} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{1,i}a_{2,j}S_iS_j - \sum_{i=1}^{3}\sum_{j=1}^{3} a_{2,i}a_{1,j}S_iS_j =$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} a_{1,i}a_{2,j}[S_i, S_j] = 0 \qquad \square$$

Claim 2. If unit vectors v_1, v_2, v_3 are mutually orthogonal, then $S_{v_1}^2 + S_{v_2}^2 + S_{v_3}^2 = 2I$.

Proof. First it is just arithmetic to check that $S_x^2 + S_y^2 + S_z^2 = 2I$. Then let $v_i = (a_{i,1}, a_{i,2}, a_{i,3})$ and $S_1 := S_x, S_2 := S_y, S_3 := S_z$. Now we have

$$S_{v_1}^2 + S_{v_2}^2 + S_{v_3}^2 = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} a_{i,j}S_j\right)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_{i,j}a_{i,k}S_jS_k =$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3} \delta_{jk}S_jS_k = S_x^2 + S_y^2 + S_z^2 = 2I \qquad \square$$

Considering the two claims above, we can reduce the task of proving the theorem to the following problem:

Find a set of vectors $V \in \mathbb{R}^3$ for which it is impossible to assign "0" and "1" (outcomes of observables S_v^2) so that within each mutually orthogonal triplet of vectors "1" is assigned to exactly two of them.

Bell in 1966 was first to prove the existence of such a set, however he did not explicitly describe it (see [6]). Just a year later Kochen and Specker came up

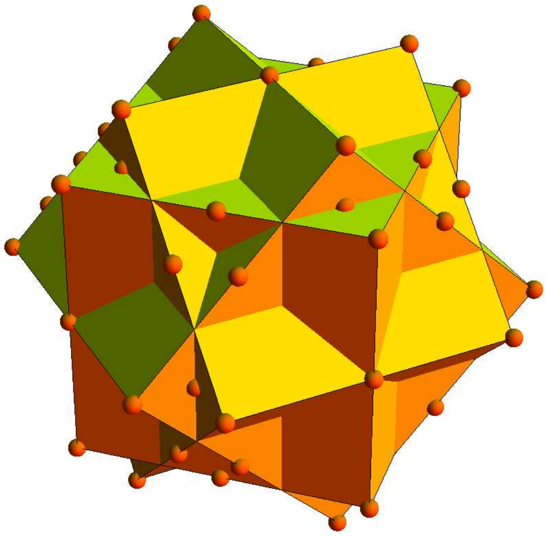

Figure 2: Three superimposed cubes. The vectors lie along the lines connecting center of the cubes with their vertices, midpoints of the edges and faces

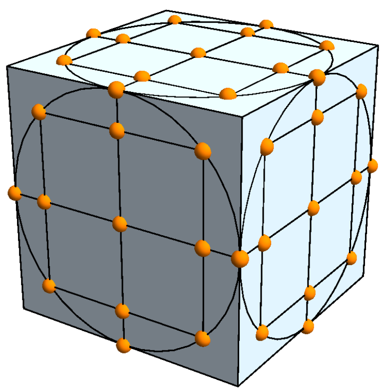

Figure 3: Peres's set of vectors satisfying the property required in the proof of Kochen-Specker theorem.

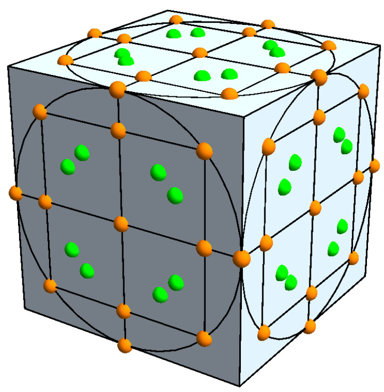

Figure 4: Peres's set of vectors completed so that every pair of orthogonal vectors has a third vector orthogonal to both of them. There are 48 green points, but half of them are redundant.

with an explicit set of 117 vectors satisfying the required property (see [5]). Yet some claim (see [1]) that most people found this proof too tedious to read. Then Peres came up with a set of 33 vectors, that enjoy a lot of symmetries making the proof much simpler (see [7] and Fig. 3). Penrose indicated that Peres's set of vectors can be interpreted as three superimposed cubes. The vectors lie along the lines connecting center of the cubes with their vertices, midpoints of the edges and faces (see Fig. 2). In both figures 3 and 2 there are $2 \cdot 33 = 66$ points, but we need only half of them, since v and $-v$ correspond to the same observable $S_v^2 = S_{-v}^2$.

Yet there is a catch in Peres's proof – I verified that in fact it *is* possible to consistently assign the outcomes for the observables corresponding to the 33 vectors in Peres's set, if restrictions are imposed only on triplets of orthogonal vectors. The catch is that he imposes restrictions also on pairs of orthogonal vectors which do not have the third orthogonal vector in his set. Since there is no functional relation like the one in Claim 2 known for *pairs* of orthogonal vectors, he implicitly imposes the restriction from Claim 2 on the pair *and* the third orthogonal vector. It means that in fact he uses more than 33 vectors. In his original proof in [7, pp. 198] one can find 8 restrictions for pairs. Unfortunately it is not enough to add the third orthogonal vector to just 8 pairs, since he also uses symmetry argument. We need to add 24 vectors, so that any pair of orthogonal vectors in the *initial* set would have a third vector orthogonal to the previous two (see Fig. 4). In fact after the addition of these 24 vectors it turns out that for *every* pair of orthogonal vectors we can find a third vector that is orthogonal to both of them. Therefore it would be fair to say that Peres's proof actually requires $33 + 24 = 57$ vectors.

2.1 Rules of the Kochen-Specker game

Let V be a set of vectors in real space from some of the proofs of 3-dimensional version of Kochen-Specker theorem. As we already saw in the case of Peres's proof, it might be necessary to take more vectors than the authors of the proofs claim.

- Verifier asks Alice to assign "0" and "1" to some mutually orthogonal triple of vectors $v_1, v_2, v_3 \in V$. He chooses one vector v_l $(l \in 1, 2, 3)$ from the triplet v_1, v_2, v_3 he gave Alice and asks Bob to assign "0" or "1" to it.
- Alice and Bob win if they satisfy two rules:
    1. *Parity rule.* Alice has to assign "1" to exactly two of the vectors.
    2. *Consistency rule.* Alice and Bob must assign the same value to the vector they have in common.

3 Quantum strategy for Kochen-Specker game

1. Alice and Bob together prepare a completely entangled two qutrit state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{1}$$

where the first qutrit belongs to Alice, but the second belongs to Bob.

2. When verifier asks Alice to assign "0" and "1" to the mutually orthogonal vectors v_1, v_2, v_3, she measures her part of the state $|\Psi\rangle$ with a POVM $|v_1\rangle\langle v_1|, |v_2\rangle\langle v_2|, |v_3\rangle\langle v_3|$. She assigns "0" to the vector to which her state collapsed to after the measurement and "1" to the other two vectors.

3. When verifier asks Bob to assign "0" and "1" to the vector $v_l \in \{v_1, v_2, v_3\}$, he measures his part of the state $|\Psi\rangle$ with a POVM $|v_l\rangle\langle v_l|, I - |v_l\rangle\langle v_l|$. Bob assigns "0" to the vector v_l if the state collapsed to $|v_l\rangle$, and "1" if otherwise.

We will now prove that the above strategy is perfect for Alice and Bob. First, it is evident that parity rule is always satisfied, since Alice always assigns "1" to exactly two vectors. Now let us see why is consistency rule always satisfied. Since vectors v_1, v_2, v_3 are mutually orthogonal they form a basis of a one qutrit state space. Let $|t\rangle = a_t^1 |v_1\rangle + a_t^2 |v_2\rangle + a_t^3 |v_3\rangle$. If we express the state $|\Psi\rangle$ in this basis, we get:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) = \sum_{t\in\{0,1,2\}} \left(\sum_{i\in\{1,2,3\}} a_t^i |v_i\rangle \right)^2 = \\ &= \sum_{t\in\{0,1,2\}} \sum_{i,j\in\{0,1,2\}} a_t^i a_t^j |v_i\rangle |v_j\rangle = \sum_{i,j\in\{0,1,2\}} \delta_{ij} |v_i\rangle |v_j\rangle = \\ &= |v_0\rangle |v_0\rangle + |v_1\rangle |v_1\rangle + |v_2\rangle |v_2\rangle,\end{aligned}$$

Therefore, if Alice's part of the state $|\Psi\rangle$ has collapsed to $v_i \in \{v_1, v_2, v_3\}$, then Bob is also left with v_i. He assigns "0" to the vector v_l if and only if $v_i = v_l$. Since Alice assigned "0" to the vector v_i, we now see that consistency rule will always be satisfied.

Note that there cannot be perfect classical strategy for Alice and Bob, as otherwise due to consistency rule we would be able to assign "0" and "1" to the vectors from the set V in a way forbidden by the 3-dimensional version of Kochen-Specker theorem.

4 Magic Square

If we are not particularly interested in proving no hidden variables theorems in three dimensions we can be content with the 4-dimensional version of Kochen-Specker theorem for which a pleasantly succinct proof is known. Also we will make a game using the set of observables considered in this proof.

Theorem 2. *(4-dimensional version of Kochen-Specker theorem)* In a Hilbert space of dimension ≥ 4 there is a set of observables for which it is impossible to assign outcomes in a way consistent with quantum mechanics formalism.

Proof. As in the proof of the 3-dimensional version of Kochen-Specker theorem we will find a set of observables that satisfy certain functional identities which cannot be satisfied by the values assigned to them. The set of observables we will use in this proof is shown in Fig. 5. It is easy to check that observables in the same row or column are mutually commuting (see Fig. 6 for multiplication table

$-I$	$-I$	$-I$	
$I\otimes Z$	$Z\otimes I$	$Z\otimes Z$	I
$X\otimes I$	$I\otimes X$	$X\otimes X$	I
$-X\otimes Z$	$-Z\otimes X$	$Y\otimes Y$	I

Figure 5: Magic square. Observables in the same row or column are mutually commuting. Observables in each row multiply to identity and to negative identity in each column.

	X	Y	Z
X	I	iZ	−iY
Y	−iZ	I	iX
Z	iY	−iX	I

Figure 6: Multiplication table of Pauli matrices.

of Pauli matrices). The functional identities that they satisfy is that observables in each row multiply to identity, whereas in each row to negative identity. From Lemma 1 we conclude that also values assigned to these observables should satisfy the same functional identities - they should multiply to +1 in each row and to -1 in each column. Yet it is not possible, since the row identities require the product of all nine values to be $(+1)^3 = 1$, whereas column identities require it to be $(-1)^3 = -1$. $\square$

4.1 Rules of the magic square game

- Verifier asks Alice to fill in the entries of some *row* of a 3×3 array with "+1" and "-1". He asks Bob to fill in some *column* of 3×3 array with "+1" and "-1".
- Alice and Bob win if they satisfy the following two rules:

  1. *Parity rule.* The parity of entries filled with "-1" has to be even for Alice (row player) and odd for Bob (column player).

2. *Consistency rule.* Alice and Bob must assign the same value for the entry on which Alice's row intersects with Bob's column.

4.2 Quantum strategy

1. Alice and Bob together prepare two copies of completely entangled quantum state:
$$|\Psi\rangle = \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)^{\otimes 2}, \tag{2}$$
where the first and the third qubit belongs to Alice, but the second and the fourth belongs to Bob.

2. When verifier asks Alice to reveal some row, she measures her qubits with the mutually commuting observables in the corresponding row of the magic square (see Fig. 5) and fills the row with outcomes of her measurement. Strategy for Bob is similar.

We will now prove that the above quantum strategy is perfect for Alice and Bob. In order to do that, we have to show that both parity and consistency rules are always satisfied. First, let us look at the parity rule. Recall that the observables in each row of the magic square multiply to identity and to negative identity in each column. Therefore, according to Lemma 1 we can conclude that the outcomes of measurements will satisfy the same functional identities. In other words, values filled in by Alice will multiply to +1, whereas entries filled in by Bob will multiply to -1. Thus, we see that parity rule will be always met.

Now let us consider consistency rule. If $\mathcal{B} = \{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ is an orthonormal basis of two qubit state space, then $\overline{\mathcal{B}} = \{\overline{|b_1\rangle}, \overline{|b_2\rangle}, \overline{|b_3\rangle}, \overline{|b_4\rangle}\}$ is another orthonormal basis of two qubit state space, where $\overline{|b_i\rangle}$ is obtained from $|b_i\rangle$ by taking the complex conjugate of each its component. If we express the Alice's part of state $|\Psi\rangle$ in basis $\mathcal{B}$ and Bob's part in basis $\overline{\mathcal{B}}$, we get

$$|\Psi\rangle = \frac{1}{4}\left(|b_1\rangle \overline{|b_1\rangle} + |b_2\rangle \overline{|b_2\rangle} + |b_3\rangle \overline{|b_3\rangle} + |b_4\rangle \overline{|b_4\rangle}\right) \tag{3}$$

Indeed, if $|b_k\rangle = a^k_{00}|00\rangle + a^k_{01}|01\rangle + a^k_{10}|10\rangle + a^k_{11}|11\rangle$, then

$$\begin{aligned}
&\frac{1}{4}\left(|b_1\rangle \overline{|b_1\rangle} + |b_2\rangle \overline{|b_2\rangle} + |b_3\rangle \overline{|b_3\rangle} + |b_4\rangle \overline{|b_4\rangle}\right) \\
&= \frac{1}{4}\sum_{k=1}^{4}\sum_{i,j\in\{0,1\}^2} a^k_i \overline{a^k_j}\,|i\rangle|j\rangle = \frac{1}{4}\sum_{i,j\in\{0,1\}^2}\delta_{i,j}\,|i\rangle|j\rangle \\
&= \frac{1}{4}(|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle)
\end{aligned}$$

which is exactly the state $|\Psi\rangle$ in (2) after rearranging the qubits.

Since we can find a full set of real eigenvectors for all Pauli matrices, it is also possible to find a basis consisting of real vectors for every row in the magic square that would diagonalize all the observables in that row. It is obvious how to choose this basis except maybe in case of last row. Therefore, we here give the basis for the last row:

$$\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Recall that we can think of measuring with commuting observables as measuring in a basis that diagonalizes these observables. Above we saw that we can diagonalize observables in each row of the magic square in a basis $\mathcal{B}$ that consists only of real vectors. Therefore bases $\mathcal{B}$ and $\overline{\mathcal{B}}$ will be the same and if Alice's part of the state $|\Psi\rangle$ collapsed to $|b_i\rangle$ after her measurement, then Bob is also left with $\overline{|b_i\rangle} = |b_i\rangle$. When Bob measures with the observable that lies on the intersection of Alice's row with his column, he of course gets the same outcome as Alice. So, we see that also consistency rule will be always satisfied and therefore the given quantum strategy is perfect for Alice and Bob.

Note that there cannot be perfect classical strategy for Alice and Bob, as otherwise due to consistency rule we would be able to fill in all the entries in the magic square in a way that they multiply to +1 in each row and to -1 in each column. But as we saw in the proof of 4-dimensional version of Kochen-Speker theorem, this is impossible.

5 Magic star

In this section we will consider another version of Kochen-Specker theorem, that is weaker than the previous ones. However, this version provides a link between Kochen-Specker theorem and Bell's theorem (see [1]). Also we will formulate another game that is based on the magic star construction use in the proof of this theorem. The proof of this theorem will be similar to the proof of 4-dimensional version of Kochen-Specker.

Theorem 3. *(8-dimensional version of Kochen-Specker theorem)* In a Hilbert space of dimension ≥ 8 there is a set of observables for which it is impossible to assign outcomes in a way consistent with quantum mechanics formalism.

Proof. The set of observables we are interested in is depicted in Fig. 7. It is easy to check that observables lying on the same edge of the magic star are mutually commuting. Functional identities satisfied by these observables are that the mutually commuting observables lying on the same edge multiply to identity, except for the horizontal edge. The commuting observables lying on the horizontal edge multiply to negative identity. Now note that it is impossible to assign each observable one of its two eigenvalues (i.e, +1 and -1) in a way that all of the above mentioned functional identities would be satisfied. This is so, since the product over all five edges $(+1)^4 \cdot (-1) = -1$ should correspond to the product of squares of all entries in a magic star, which of course is always a positive number. □

5.1 Rules of the magic star game

- Verifier asks Alice to fill in the four entries along some edge of magic star with "+1" and "-1". He asks the same of Bob.

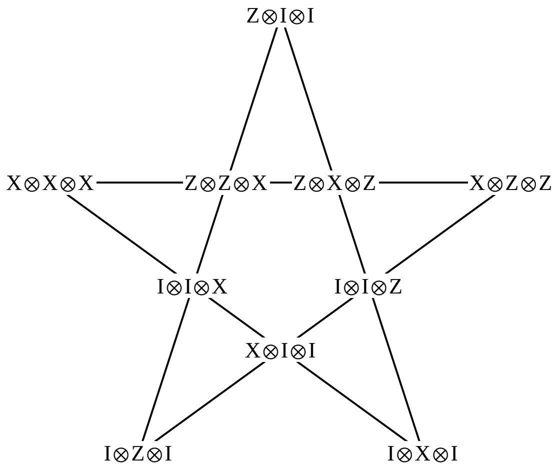

Figure 7: Magic star. Observables lying on the same edge of the star are mutually commuting. Observables along every edge of the star except the horizontal one multiply to identity and to negative identity along the horizontal edge.

- Alice and Bob win if they satisfy the following two rules
  1. *Parity rule.* The parity of entries filled with "-1" has to be even for every edge except for the horizontal one. The parity of entries filled with "-1" has to be odd for the horizontal edge.
  2. *Consistency rule.* Alice and Bob must assign the same value for the entry (or entries) on which Alice's edge intersects with Bob's column.

5.2 Quantum strategy

1. Alice and Bob together prepare three copies of completely entangled quantum state:
$$|\Psi\rangle = \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)^{\otimes 3}, \tag{4}$$
where the first, the third and the fifth qubit belongs to Alice, but the second, the fourth and the sixth belongs to Bob.

2. When verifier asks Alice to reveal some edge, she measures her qubits with the mutually commuting observables on the corresponding edge of the magic star (see Fig. 7) and fills the entries with outcomes of her measurement. Strategy for Bob is similar.

Strategy for magic star game is similar to that of magic square game, so the proof that this strategy is perfect will also be similar. We need to show that both parity and consistency rules are always satisfied. Since observables lying on the edges other than horizontal multiply to identity and observables on the horizontal edge multiply to negative identity, according to Lemma 1 we can conclude that parity rule will be always satisfied.

Now let us examine consistency rule. If $\mathcal{B} = \{|b_i\rangle\}_{i=1}^{8}$ is an orthonormal basis of three qubit state space, then $\overline{\mathcal{B}} = \{\overline{|b_i\rangle}\}_{i=1}^{8}$ is another orthonormal basis of two qubit state space, where $\overline{|b_i\rangle}$ is obtained from $|b_i\rangle$ by taking the complex conjugate of each its component. If we express the Alice's part of state $|\Psi\rangle$ in basis $\mathcal{B}$ and Bob's part in basis $\overline{\mathcal{B}}$, we obtain an expression similar to (3):

$$|\Psi\rangle = \frac{1}{8}\sum_{i=1}^{8} |b_i\rangle\, \overline{|b_i\rangle}$$

As in case of magic square we can diagonalize all the observables on each edge of the magic star in a basis $\mathcal{B}$ consisting only of real vectors, since it is possible to find a full set of real eigenvectors for every Pauli matrix. It is evident, how to find this basis except maybe for the horizontal edge. Therefore, we here give the basis that diagonalize the observables on the horizontal edge:

$$\begin{pmatrix}-1\\1\\1\\1\\1\\1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\1\\1\\1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\1\\1\\-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\-1\\-1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\\1\\-1\\-1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\1\\-1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\1\\-1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\-1\\1\\-1\\-1\\-1\end{pmatrix},$$

When Alice measures with commuting observables lying on some edge of the magic star, we can think of it as a measurement in basis $\mathcal{B} = \{|b_i\rangle\}_{i=1}^{8}$ consisting only of real vectors. Therefore, if Alice's state after her measurement collapses to $|b_i\rangle$ Bob is also left with $\overline{|b_i\rangle} = |b_i\rangle$, since $\mathcal{B} = \overline{\mathcal{B}}$. When Bob measures with the observable that lies on the intersection of Alice's row with his column, he of course gets the same outcome as Alice. So, we see that also consistency rule will be always satisfied and therefore the given quantum strategy is perfect for Alice and Bob.

Note that as in case of magic square there cannot be perfect classical strategy for Alice and Bob, as otherwise due to consistency rule we would be able to fill in all the entries in the magic star in a way forbidden by 8-dimensional version of Kochen-Specker theorem.

6 Acknowledgements

This essay was inspired by the captivating paper [1] by John Mermin, in which he describes several versions of Kochen-Specker theorem. The Kochen-Specker game comes from paper [2]. The Magic square game comes from Aravind's paper [3], while the magic star game I made myself using the ideas employed in magic square game. At last but not least I want to thank Maris Ozols for nice pictures, he made for me using *Mathematica* and for helping me to run computer simulations with Peres's set of 33 vectors.

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